LECTURE 26 MEAN VALUE THEOREM

Before we continue with the consequence of the mean value theorem, we go through a quick example showcasing some common sense about projectile.

Example. The range R of a projectile launched on the ground, i.e. horizontal distance travelled, satisfies

$$R = \frac{v_0^2}{g} \sin(2\theta)$$

where $v_0 > 0$ is the known initial launch velocity, g the gravitational acceleration, and θ the launch angle. Show that to maximize range, we should launch at $\theta = \frac{\pi}{4}$.

Solution. Now, launch angle goes between $[0, \pi]$. We maximize R on this interval. First, we find the critical points that satisfy

$$\frac{dR}{d\theta} = \frac{v_0^2}{g} 2\cos(2\theta) = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}.$$

Then, we evaluate the critical point and the endpoints.

$$R(0) = 0$$
$$R(\pi) = 0$$
$$R\left(\frac{\pi}{4}\right) = \frac{v_0^2}{g} > 0$$

which means $\theta = \frac{\pi}{4}$ maximizes R.

Now, we continue with the consequence of the mean value theorem. We restate the theorem for recall.

Theorem. *(Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark. Note that differentiability is NOT needed at $x = a$ and $x = b$, only left and right continuity respectively.

Another interpretation is that the average rate of change must be equal to the instantaneous rate of change somewhere between. In fact, many real world problems can lead to nice conclusions using the mean value theorem. We see its usage in detecting speeding cars!

Example. This very fact also explains how electronic cameras can decide whether a car has sped or not. Consider two cameras placed 1 mile apart on a highway with speed limit 60 miles/hour. Now, consider time t as our independent variable and car position $x(t)$ as our dependent variable. Denote t_1 and t_2 as the time the car reaches camera 1 and 2 respectively. Suppose x is a continuous function of t on $[t_1, t_2]$ and differentiable on (t_1, t_2), then we have by mean value theorem that there exists some time $s \in (t_1, t_2)$ such that

$$x'(s) = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{1}{t_2 - t_1}.$$

The camera will have access to the time difference $t_2 - t_1$. Therefore, it is able to compute $\frac{1}{t_2 - t_1}$ and thus knows there is definitely one point at which the car is travelling at $\frac{1}{t_2 - t_1}$. Now, suppose $t_2 - t_1$ is measured in seconds, we convert it to hours by writing $\frac{t_2 - t_1}{3600}$. Therefore, if the cameras detect

$$\frac{3600}{t_2 - t_1} > 60,$$

the car is guaranteed to have sped at some point between the two cameras.

The next result is a direct consequence of the mean value theorem, though the result itself may seem trivial, yet it is not.

Corollary. *If $f'(x) = 0$ at every point x of an open interval (a,b), then $f(x) = C$ for all $x \in (a,b)$, where C is a constant.*

Proof. Note that f satisfies the hypothesis of the mean value theorem. We wish to show $f(x_1) = f(x_2)$ with $x_1 < x_2$ for <u>any</u> two points on (a,b).

Indeed, by the mean value theorem, for any two points $x_1 < x_2$ on (a,b), there must be a point $y \in (x_1, x_2)$ such that

$$0 = f'(y) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_2) = f(x_1).$$

□

When two functions have the same derivative, can we relate the two functions themselves?

Corollary. *If $f'(x) = g'(x)$ at every point $x \in (a,b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a,b)$.*

Proof. Define $h(x) = f(x) - g(x)$ and we note $h'(x) = 0$. Then by the previous corollary, we are done. □

Example. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

Solution. We assume that we have no integration knowledge but only the two corollaries above. We don't know what $f(x)$ looks like. The idea is to cook up a function $g(x)$ such that $g'(x) = 2$ and use the last corollary. We find that $g(x) = 2x$ will work. Thus, we find that $f'(x) = g'(x)$ for all x. Therefore,

$$f(x) = g(x) + C = 2x + C.$$

We can solve for C by using $f(0) = 5$ which shows that $C = 5$. Thus $f(x) = 2x + 5$.

Next, we go on to Section 4.3 and study a particular type of functions useful in determining the behaviour of more complicated functions.

Definition. We say that f is **monotone increasing** on an interval I if we have $f(x_1) < f(x_2)$ (resp. **monotone decreasing** with $>$), for every $x_1 < x_2$ in I.

Corollary. *If f is continuous on $[a,b]$ and differentiable on (a,b), and if $f'(x) > 0$ for every $x \in (a,b)$, then f is **monotone increasing** (resp. < 0, **monotone decreasing**).*

Proof. This again, is a direct consequence of the mean value theorem. For any two points $x_1 < x_2$ on (a,b), there is some point c such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(x) > 0$ and also $x_2 > x_1$, the RHS is positive, which implies $f(x_2) - f(x_1) > 0 \implies f(x_1) < f(x_2)$, namely, f is **monotone increasing**. The proof for monotone decreasing is similar. □

Example. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution. We find that the critical points satisfy

$$0 = f'(x) = 3x^2 - 12 \implies x = \pm 2.$$

We then create nonoverlapping intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$. We plug in convenient values from each interval to check the sign of f

$$\begin{aligned} f'(-3) &= 3(-3)^2 - 12 = 15 > 0 \\ f'(0) &= -12 < 0 \\ f'(3) &= 3(3)^2 - 12 = 15 > 0 \end{aligned}$$

We did first derivative test already. At a critical point $x = c$,

1. if f' changes from negative to positive at c, then f has a local minimum at $x = c$.
2. if f' changes from positive to negative at c, then f has a local maximum at $x = c$.
3. if f' does not change sign (that is, f' has the same sign on both sides of c), then f has no local extremum at c.